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# Quantum phase transition of entanglement in Heisenberg spin model with quantized field

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**Abstract.** In this paper, we investigate the entanglement of a two-spin system with Heisenberg exchange interaction in a quantized field. The pairwise entanglement between bipartite subsystems is obtained. It is shown that the entanglement exhibits a quantum phase transition due to the variation of exchang coupling. Phase diagrams are obtained explicitly. The analogy of the quantum phase transition compared to the case under a classical field are addressed.

**PACS.** 03.65.Ud Entanglement and quantum nonlocality (e.g. EPR paradox, Bells inequalities, GHZ states, etc.) – 75.10.Jm Quantized spin models – 03.67.Lx Quantum computation

## 1 Introduction

Entanglement is a correlation of quantum nature presenting in quantum systems exclusively. It plays a central role in quantum information and computation [1] and recently has been considered as an important resource of quantum information manipulation [2]. Therefore, one of the main tasks of quantum information theory is to quantify the entanglement and the quantum correlations between quantum states [3–6].

The spin system with Heisenberg exchange interaction is a typical quantum system and the entanglement properties associated with it have been extensively investigated recently [7–13]. The entanglement of Heisenberg spin system driven by external fields has been studied to date mainly in a semi-classical context, in which the field is treated classically, in the research literature. However, many effects in quantum optics (i.e., quantum jumps, collapses and revivals of the Rabi oscillations) can be explained only with a complete description involving field quantization [14]. Moreover, in quantum mechanics, several interesting effects, such as spontaneous emission and Lamb shift, are observed due to the interaction of quantum systems with the vacuum in the full quantum theory [15]. This vacuum effect can be explained by the full-quantized theory only. Recently, studies on the quantum peculiarity of spin system with a quantized external field attract extensive attention in quantum theory community. For example, Fuentes-Guridi et al. [16], Carollo et al. [17] and Wang et al. [18] have demonstrated the effect of a driven field quantization on the geometric phase in spin systems.

It is an interesting quantum phenomenon that the entanglement shares many features with quantum phase transition (QPT) ([19–26], and references therein). QPTs in an interactive many-body system are the structural changes in the properties of the ground state. They occur at zero temperature and, thus, are purely driven by quantum fluctuations. The associated level crossings, in many cases, lead to the presence of non-analyticities in the energy spectrum. Therefore, the knowledge about the entanglement, the nonlocal correlation in quantum systems, is believed as the key to understand QPTs [19]. In other words, fully understanding of the multipartite entanglement does a great help to explain the global correlation in QPTs.

Identification of the characteristic of a bipartite system, the simplest many-body model, is of great scientific interests and such simple system provides us new paths to explore the secrets of QPTs without the difficulties encountered for other QPT phenomena where more complex many-body interactions involve, especially when an external field presents. In this paper, we investigate the pairwise entanglement between two spins under Heisenberg exchange interactions driven by a quantized external field and examine the related QPT as well. It is found that, in the Ising model, when the particle-field coupling is on resonance and the coupling constant $\Omega$ tends to be zero, the entanglement appears as a mutation from 1 to 0 which

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$$H=\begin{pmatrix} \Delta+\nu\left(n+1\right)+J_z & \Omega\sqrt{n+1} & \Omega\sqrt{n+1} & 0\\ \Omega\sqrt{n+1} & \nu\left(n+1\right)-J_z & 2J & \Omega\sqrt{n+2}\\ \Omega\sqrt{n+1} & 2J & \nu\left(n+1\right)-J_z & \Omega\sqrt{n+2}\\ 0 & \Omega\sqrt{n+2} & \Omega\sqrt{n+2} & -\Delta+\nu\left(n+1\right)+J_z \end{pmatrix}. \tag{2}$$

$$B=\begin{pmatrix} -\sqrt{\frac{n+2}{2n+3}} & 0 & 0 & \sqrt{\frac{n+1}{2n+3}}\\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\ \Omega\sqrt{\frac{n+1}{\zeta[\zeta+(J_z-J)]}} & -\frac{1}{2}\sqrt{\frac{\zeta+(J_z-J)}{\zeta}} & -\frac{1}{2}\sqrt{\frac{\zeta+(J_z-J)}{\zeta}} & \Omega\sqrt{\frac{n+2}{\zeta[\zeta+(J_z-J)]}}\\ \Omega\sqrt{\frac{n+1}{\zeta[\zeta-(J_z-J)]}} & \frac{1}{2}\sqrt{\frac{\zeta-(J_z-J)}{\zeta}} & \frac{1}{2}\sqrt{\frac{\zeta-(J_z-J)}{\zeta}} & \Omega\sqrt{\frac{n+2}{\zeta[\zeta-(J_z-J)]}} \end{pmatrix}. \tag{5}$$

indicates that a QPT takes place. In the semi-classical case, a QPT occurs when the angle of magnetic field $\theta=\pi/2$ and field intensity $B\to 0$ [10]. When the particle-field coupling is out of resonance, a case with more general Heisenberg exchange interactions was studied and a rich structure of the entanglement phase diagrams was observed.

This paper is organized as the follows: Section 2 is the definition of the Hamiltonian and the calculation approach of the pairwise entanglement in a bipartite system. Then, in Sections 3 and 4, the analytical and numerical results of the entanglement in the ground state are presented and discussed, respectively. Finally, we conclude briefly in Section 5.

## 2 Model and theoretical method

The composite system we studied consists of two spins with Heisenberg exchange interaction under the presence of a quantized field of one mode. Without the exchange interaction, this model can be also used to describe a two-level atom coupled with a single-mode electromagnetic field via the electric-dipole interaction. In the rotating-wave approximation (RWA), the Hamiltonian reads

$$\begin{aligned} H=&\nu a^\dagger a+\frac{\omega}{2}(\sigma_1^z+\sigma_2^z)+\Omega[a(\sigma_1^++\sigma_2^+)+a^\dagger(\sigma_1^-+\sigma_2^-)]\\ &+J_x\sigma_1^x\sigma_2^x+J_y\sigma_1^y\sigma_2^y+J_z\sigma_1^z\sigma_2^z, \end{aligned} \tag{1}$$

where $\omega$ is the transition frequency between the eigenstates of the spin-$\frac{1}{2}$ system (the natural unit $\hbar=1$ is used throughout the paper), $\nu$ is the frequency of the field described in terms of the creation and annihilation operators $a^\dagger$ and $a$. $\Omega$ is the coupling constant between the field and particles known as the Rabi frequency. $J_x$, $J_y$and $J_z$ are exchange-coupling constants between two spin-$\frac{1}{2}$ particles and $\sigma_i^\pm=\sigma_i^x\pm\sigma_i^y$ as $\sigma_i^x$, $\sigma_i^y$ and $\sigma_i^z$ are common Pauli spin matrix with subscripts $i=1$ and 2 describing two different spins respectively. All the spin-$\frac{1}{2}$ particles are assumed to be coupled to the field with the same coupling constant. When $J_x=J_y=J$ the Hamiltonian is known as the $XXZ$ model, while it reduces to the $XXX$ model if $J_z=J$ ($XX$ model for $J_z=0$) and Ising model if $J=0$, $J_z\neq 0$. We set $\Delta=\omega-\nu$ denoting the detuning. In what follows, we calculate the entanglement for the case of resonance, i.e., $\Delta=0$, analytically; with regard to the case when $\Delta\neq 0$, although there are the analytical solutions to it, the results in form of graphical figures are presented instead for clarity and simplification.

Given boson number of the field, the corresponding Hilbert space is spanned by four basic state vectors $|n,\uparrow_1,\uparrow_2\rangle$, $|n+1,\uparrow_1,\downarrow_2\rangle$, $|n+1,\downarrow_1,\uparrow_2\rangle$ and $|n+2,\downarrow_1,\downarrow_2\rangle$ where $|n\rangle$ is field quantum state in Fock representation and $|\downarrow\rangle$, $|\uparrow\rangle$ describe spin up and down states, respectively. For simplicity, the four basic state vectors are expressed as $|\Phi_i\rangle$ , $i=1,2,3$ and 4, and the spin vectors $|\uparrow_1,\uparrow_2\rangle$, $|\uparrow_1,\downarrow_2\rangle$, $|\downarrow_1,\uparrow_2\rangle$ and $|\downarrow_1,\downarrow_2\rangle$ are denoted by $|\phi_i\rangle$ , $i=1,2,3$ and 4. The Hamiltonian of $XXZ$ model can be expressed as

*see equation (2) above*

For the resonance case ($\Delta=0$), the eigenvalues of Hamiltonian (2), by neglecting the unimportant constant term $\nu\left(n+1\right)$ in all eigenvalues, are obtained as

$$E_1=J_z,\ E_2=-J_z-2J,\ E_3=J-\zeta,\ E_4=J+\zeta, \tag{3}$$

with $\zeta=\sqrt{\left(J_z-J\right)^2+2\left(3+2n\right)\Omega^2}$ and the corresponding eigenstates are,

$$|\psi_i\rangle=\sum_{j=1}^{4}b_{i,j}\,|\Phi_j\rangle\,, \tag{4}$$

where the coefficients $b_{i,j}$ are elements of matrix $B$,

*see equation (5) above*

It is straightforward to verify the orthogonal relation that

$$\sum_{k=1}^{4}b_{k,i}b_{k,j}^{*}=\delta_{i,j}. \tag{6}$$

The standard measure of pairwise entanglement for pure state of a bipartite system is the Von Neumann entropy of either of the two subsystems [3]. If the density matrix $\rho_{12}$ of a composite system composed of subsystems 1 and 2, the reduced density operator for subsystem 1 is $\rho_1=\mathrm{Tr}_2\left(\rho_{12}\right)$, and the entropy of entanglement would be

$$E\left(\rho_{12}\right)=-\mathrm{Tr}\left(\rho_1\log_2\rho_1\right). \tag{7}$$

$$H = \begin{pmatrix} \Delta + \nu(n+1) + J_z & 0 & 0 & 0 \\ 0 & \nu(n+1) - J_z & 2J & 0 \\ 0 & 2J & \nu(n+1) - J_z & 0 \\ 0 & 0 & 0 & -\Delta + \nu(n+1) + J_z \end{pmatrix}, \quad (10)$$

In our case, the density operator of two spins for a given phonon number $n$ is

$$\rho_{12} = \mathrm{Tr}_{field}(\rho) = \langle n|\rho|n\rangle + \langle n+1|\rho|n+1\rangle + \langle n+2|\rho|n+2\rangle, \quad (8)$$

here $\rho$ denotes the density operator of the whole system consisting of spin and field. Generally speaking, $\rho_{12}$ is a mixed state, so we cannot directly use the entropy as a measure of the entanglement. We introduce the entanglement of formation, which is the common measurement of entanglement for a mixed state with the general definition as the minimum average of entanglement of an ensemble of pure states [6]. There are several specific definitions of the entanglement of formation in a bipartite system and here we adopt the Wootters's Concurrence [6], which is defined by

$$C = \max\left\{0, 2\max\{\lambda_i\} - \sum \lambda_i\right\}, \quad (9)$$

where $\lambda_i$s denote the square roots of the eigenvalues of matrix $R$,

$$R = \rho_{12}(\sigma^y \otimes \sigma^y)\rho_{12}^*(\sigma^y \otimes \sigma^y)$$

where $\sigma^y$ is Pauli matrix, $\rho_{12}^*$ is conjugate of $\rho_{12}$. Note that, when spin-field coupling constant vanishes $\Omega = 0$, the density operator of two spins $\rho_{12}$ describes a pure state as the Hilbert spaces of spin and field are separated. In this case, the Hamiltonian becomes

*see equation (10) above*

which is expressed on basis $\{|\phi_i\rangle\}$, not $\{|\Phi_i\rangle\}$. Then the eigenvalues and the corresponding eigenstates of the Hamiltonian are:

$$E_1^0 = J_z,\ E_2^0 = -J_z - 2J,\ E_3^0 = -J_z + 2J,\ E_4^0 = J_z, \quad (11)$$

$$\begin{aligned} |\psi_1^0\rangle &= |\phi_1\rangle, \\ |\psi_2^0\rangle &= \frac{1}{\sqrt{2}}(|\phi_2\rangle - |\phi_3\rangle), \\ |\psi_3^0\rangle &= \frac{1}{\sqrt{2}}(|\phi_2\rangle + |\phi_3\rangle), \\ |\psi_4^0\rangle &= |\phi_4\rangle, \end{aligned} \quad (12)$$

where $|\psi_2^0\rangle$ and $|\psi_3^0\rangle$ are the Bell bases. The entropy of entanglement of these eigenstates is known as $E(|\psi_1^0\rangle) = E(|\psi_4^0\rangle) = 0$ and $E(|\psi_2^0\rangle) = E(|\psi_3^0\rangle) = 1$. As a exception, when $J = 0$ (*Ising model*), the Hamiltonian is diagonal in this case and the eigenstates are degenerated ($E_1^0 = E_4^0 = 1, E_2^0 = E_3^0 = -1$), and simplified to $|\psi_i^0\rangle = |\phi_i\rangle$ which are not entangled states any more.

## 3 Entanglement with antiferromagnetic coupling

The entanglement depends sensitively on the anisotropy of the exchange coupling. To clarify this, we examine a variety of anisotropy regions of exchange coupling.

### 3.1 $J = 0, J_z > 0$

This is known as Ising model. Without the external field the ground state is not entangled, as it should be, while with the quantized field i.e., $\Omega \neq 0$, eigenvalues become

$$E_1 = J_z,\ E_2 = -J_z,\ E_3 = -\zeta,\ E_4 = \zeta, \quad (13)$$

with $\zeta = \sqrt{J_z{}^2 + 2(3+2n)\Omega^2} > J_z$, the ground state is $|\psi_3\rangle$, so that density operator of ground state is

$$\rho = |\psi_3\rangle\langle\psi_3| = \sum_{i,j=1}^{4} b_{3,i}b_{3,j}^* |\Phi_i\rangle\langle\Phi_j|. \quad (14)$$

The reduced density operator of two spins is

$$\rho_{12} = \sum_{i=1}^{4} |b_{3,i}|^2 |\phi_i\rangle\langle\phi_i| + b_{3,2}b_{3,3}^* |\phi_2\rangle\langle\phi_3| + b_{3,3}b_{3,2}^* |\phi_3\rangle\langle\phi_2|, \quad (15)$$

from which the concurrence of ground state is obtained as

$$C = \max\{0, 2|b_{3,2}b_{3,3}| - 2|b_{3,1}b_{3,4}|\} = \frac{\zeta + J_z}{2\zeta} - \frac{\zeta - J_z}{\zeta}\frac{\sqrt{(n+1)(n+2)}}{3+2n}. \quad (16)$$

It increases with the decrease of spin-field coupling $\Omega$ and tends to equal 1 when $\Omega$ approaches 0. However we have shown that the eigenstates of Hamiltonian are not entangled for $\Omega = 0$, therefore a mutation of entanglement from 1 to 0 takes place at the critical value of the coupling constant $\Omega_c = 0$ shown in Figure 1a, which can be considered as a QPT. In order to show the entanglement behavior at $\Omega = 0$, the figure is presented symmetrically with respect to $\Omega$, (extending to the negative value) in virtue of the $\Omega^2$-dependence of the entanglement, although the $\Omega$ is non-negative. The numerical result with non-zero detuning $\Delta$ is given in Figure 2a, which is quite analogous to the coupling constant dependence of the entanglement of two spins driven by a classical field [10], where entanglement is shown as a function of the magnetic field components, $B_x$ and $B_z$.

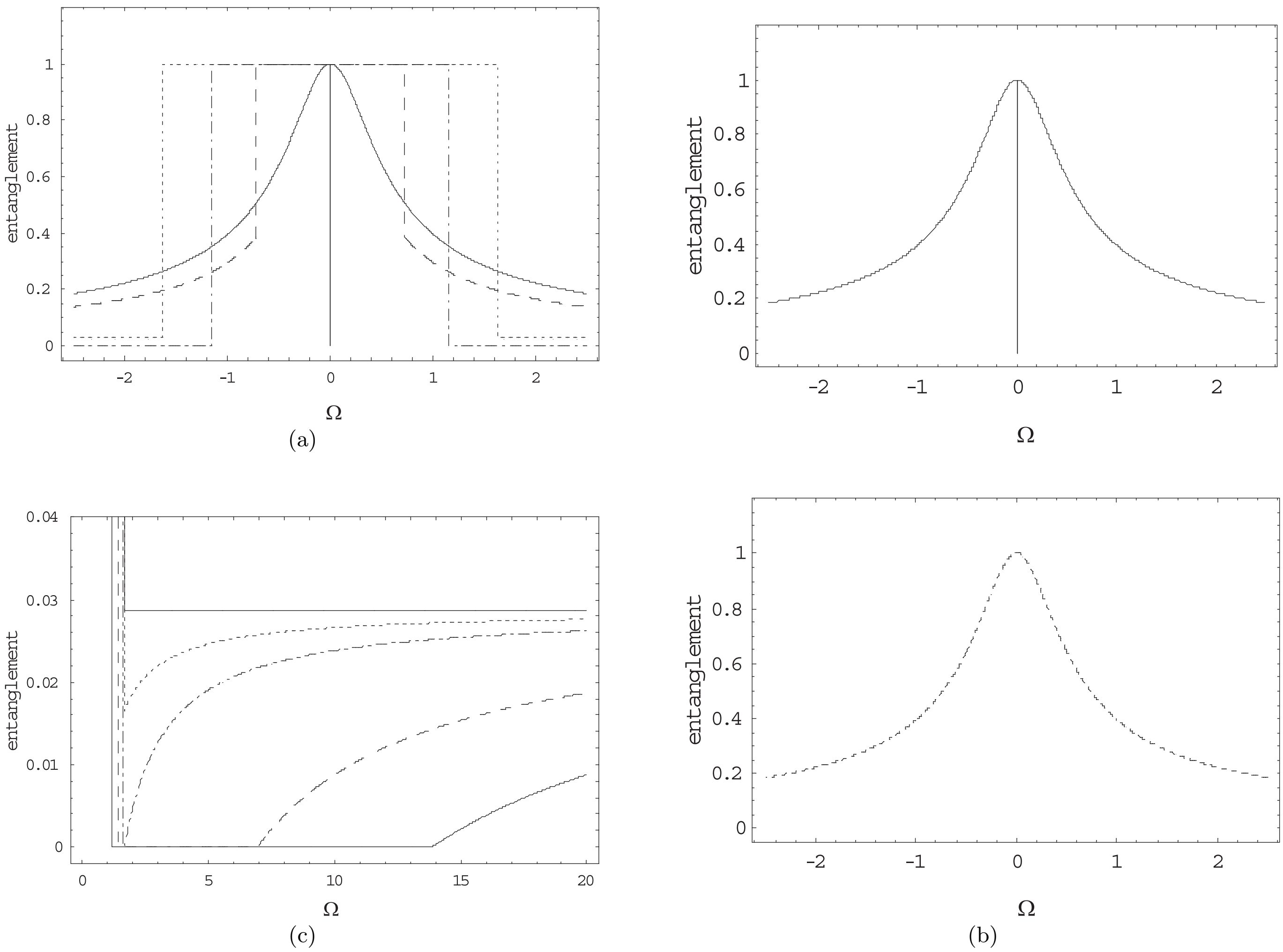


**Fig. 1.** (a) Concurrence of the antiferromagnetic $XXZ$ model versus coupling constant $\Omega$ for different $J_z$ and $J$: $J = 0$ and $J_z = 1$ (solid line), $J = 0.3$ and $J_z = 1$ (dashed line), $J = 1$ and $J_z = 0$ (dash-dot-line), $J = J_z = 1$ (doted line). (b) Comparison of two cases for $J = 0$ (up) and $J = 0.001$ (down) for $J_z = 1$. (c) Concurrence of the antiferromagnetic $XXZ$ model versus coupling constant $\Omega$ for $J_z \le J$: $J = 1$ and $J_z = 0$ (solid line down), $J = 1$ and $J_z = 0.5$ (dashed line), $J = 1$ and $J_z = 0.88$ (dash-dot-line), $J = 1$ and $J_z = 0.95$ (doted line), $J = J_z = 1$ (solid line up).

### 3.2 $\mathbf{J_z > J > 0}$

For the case of vanishing magnetic field i.e. $\Omega = 0$, it is derived from equation (11) that ground state is $\left|\psi_2^0\right\rangle$ and the entropy of entanglement is of a maximum value, 1. With the driving field ($\Omega \neq 0$), $E_1 > E_2$ and $E_4 > E_3$ is obtained from equation (3). Given $E_2 = E_3$, a critical value of the Rabi frequency is obtained as

$$\Omega_c = 2\sqrt{\frac{J\left(J + J_z\right)}{2n + 3}}. \tag{17}$$

And, correspondingly,

$$\zeta_c = \sqrt{\left(J_z - J\right)^2 + 8J\left(J + J_z\right)}. \tag{18}$$

For the case $\Omega > \Omega_c$, we obtain $E_3 < E_2$, the ground state is $|\psi_3\rangle$. The concurrence of ground state is

$$C\left(|\psi_3\rangle\right) = \frac{\zeta + \left(J_z - J\right)}{2\zeta} - \frac{\zeta - \left(J_z - J\right)}{\zeta}\frac{\sqrt{\left(n+1\right)\left(n+2\right)}}{3 + 2n} \tag{19}$$

which decreases as $\Omega$ increasing and approaches the asymptotic value $C_a = 1/2 - \sqrt{\left(n+1\right)\left(n+2\right)}/(3+2n)$. When $\Omega \to \Omega_c$, we have

$$C\left(|\psi_3\rangle\right) \to C\left(\Omega_c\right) = \frac{\zeta_c + \left(J_z - J\right)}{2\zeta_c} - \frac{\zeta_c - \left(J_z - J\right)}{\zeta_c}\frac{\sqrt{\left(n+1\right)\left(n+2\right)}}{3 + 2n} < 1.$$

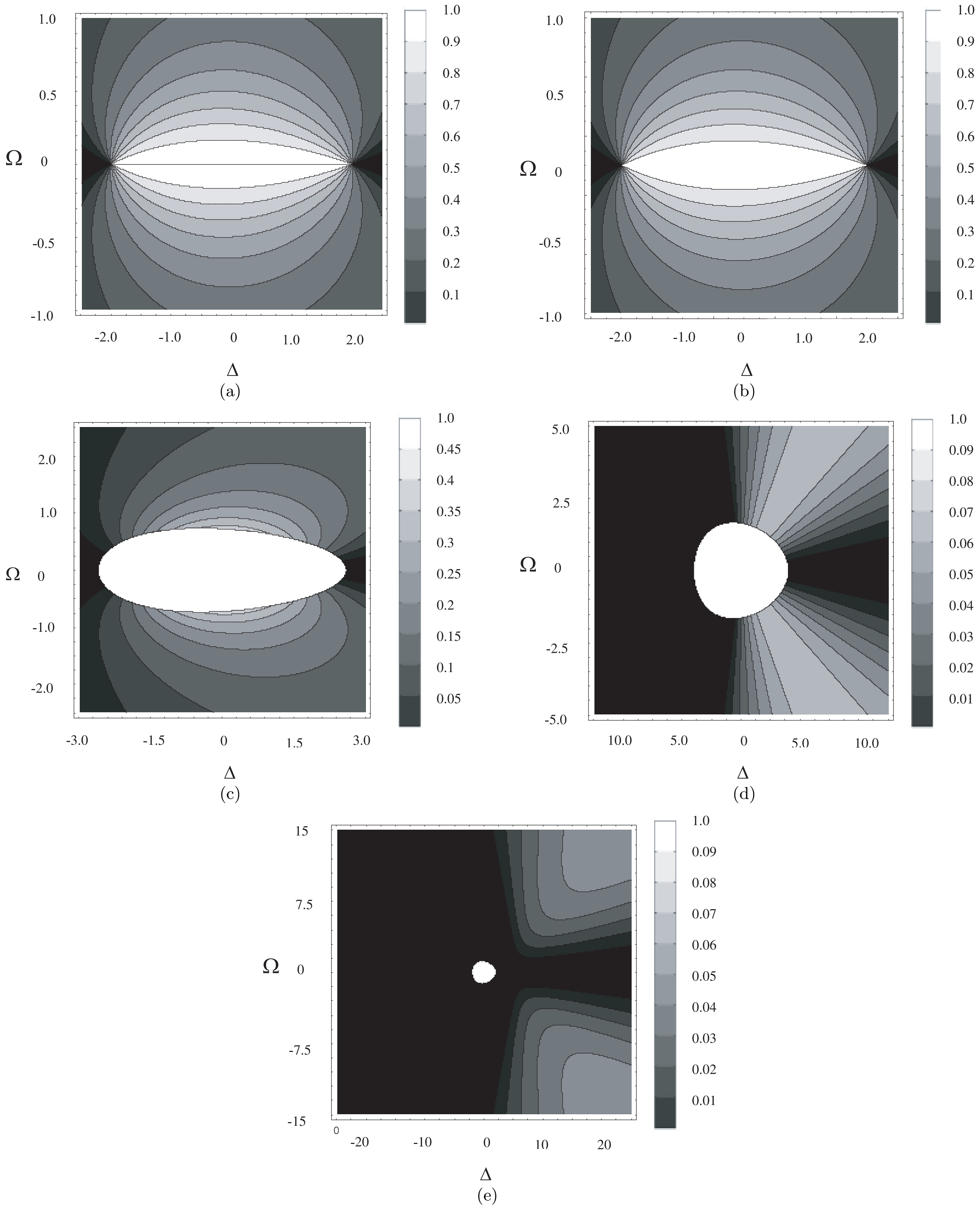


**Fig. 2.** Concurrence of the antiferromagnetic *XXZ* model versus detuning $\Delta$ ($x$-coordinate) and coupling constant $\Omega$ ($y$-coordinate) for different $J$ and $J_z$. Figures a–e correspond to $J = 0$ and $J_z = 1$, $J = 0.001$ and $J_z = 1$, $J = 0.3$ and $J_z = 1$, $J = J_z = 1$, $J = 1$ and $J_z = 0$.

If $\Omega < \Omega_c$, then, $E_2 < E_3$ and the ground state is $|\psi_2\rangle$. The concurrence becomes

$$C = \max\{0, 2|b_{2,2}b_{2,3}| - 2|b_{2,1}b_{2,4}|\} = 1. \quad (20)$$

A mutation of concurrence from 1 to $C(\Omega_c)$ takes place at the critical value of Rabi frequency $\Omega = \Omega_c$, where $\Omega_c$ is a critical point on which QPT occurs.

The concurrence as function of $\Omega$¸ for different exchange coupling constants $J$ and $J_z$ shown in Figure 1a. When $\Omega = 0$, it is shown that the QPT occur in the case of $J = 0$ discussed previously, while, a very small coupling constant $J$ can lead to absence (or disappearance) of QPT. Figure 1b shows the comparison between the two cases. The numerical results with non-zero detuning $\Delta$ for different $J$ and $J_z$ are presented in Figure 2. Of course, the detuning is meaningless when the coupling constant $\Omega = 0$, while a mutation of entanglement from 1 to 0 takes place, when there is external field with $\Omega$ close to 0, at the detuning $\Delta = 2.0$ shown in Figures 2a and 2b.

### 3.3 $J_z = J > 0$

This case is known as the *XXX* model. Again the entropy of entanglement of the ground state is maximum (**1** here) without the driven field. When $\Omega \neq 0$, we find a critical value of the spin-field coupling that

$$\Omega_c = 2J\sqrt{\frac{2}{2n+3}}, \quad (21)$$

so that the concurrence of ground state, which is unit 1 when $\Omega < \Omega_c$, jumps to a value $C_a = 1/2 - \sqrt{(n+1)(n+2)}/(3+2n)$ (which is independent of the exchange coupling constant) when $\Omega > \Omega_c$ (see Figs. 1a and 1c, where field quantum number $n = 0$, $C_a = 1/2 - \sqrt{2}/3 \cong 0.0286$).

### 3.4 $J > J_z \geq 0$

Without the external field ($\Omega = 0$) ground state is $|\psi_2^0\rangle$ and the entropy of entanglement is **1**, which is the same as previous cases.

However, when $\Omega \neq 0$, a more complex structure has been observed. If $\Omega < \Omega_c$ ($\Omega_c$ given in Eq. (17)), the ground state is $|\psi_2\rangle$ with concurrence $C = 1$. If $\Omega > \Omega_c$, the ground state is $|\psi_3\rangle$, and the concurrence is

$$C = \max\left\{0, \frac{\zeta - (J - J_z)}{2\zeta} - \frac{\zeta + (J - J_z)}{\zeta}\frac{\sqrt{(n+1)(n+2)}}{3+2n}\right\}. \quad (22)$$

According to equation (22), there is another critical point,

$$\Omega_c^{,} = \frac{2\sqrt{(J_z - J)^2\left[4(n+1)(n+2) + \sqrt{n+1}\sqrt{n+2}(2n+3)^2\right]}}{[7+4n(3+n)]\sqrt{(2n+3)}},$$

coming into the consideration. When $\Omega < \Omega_c^{,}$, the second term in the brace is less than zero, therefore, there are two sub-cases for the concurrence dependent on $\Omega$: (i) $\Omega_c < \Omega_c^{,}$ and (ii) $\Omega_c > \Omega_c^{,}$. In the first sub-case, $C = 1$ locates in the region $0 \leq \Omega < \Omega_c$. When $\Omega < \Omega_c < \Omega_c^{,}$, the concurrence vanishes and at the critical point $\Omega = \Omega_c$, a mutation of the entanglement from 1 to 0 takes place. When $\Omega > \Omega_c^{,}$, concurrence increases with $\Omega$¸ and approaches the asymptotic value $C_a$. In the second sub-case, the concurrence is 1 in the region $0 \leq \Omega < \Omega_c$ as well, and jumps from 1 to

$$C_c = \frac{\zeta_c - (J - J_z)}{2\zeta_c} - \frac{\zeta_c + (J - J_z)}{\zeta_c}\frac{\sqrt{(n+1)(n+2)}}{3+2n}$$

at the critical point $\Omega = \Omega_c$. When $\Omega > \Omega_c$, the concurrence $C$ increasing with $\Omega$ from $C_c$ approaches the asymptotic value $C_a$.

When $J \geq J_z$, the concurrence for the general *XXZ* model is a function of $\Omega$ in the resonance shown as Figure 1c in which *XXX* model ($J = J_z$) and *XX* model ($J_z = 0$) are two special cases. Figures 2b to 2e show the numerical results of concurrence with detuning $\Delta$ for different $J_z$ and $J$. These figures are the phase diagrams of the system with regard to the entanglement, in the white regions of which the concurrence is 1. In Figures 2c to 2e a mutation of the entanglement occurs on the boundary. This can be considered as a quantum phase transition occurring on the boundary.

## 4 Entanglement with ferromagnetic coupling

The dependence of the entanglement on the anisotropy of the exchange coupling in Ferromagnetic is somehow simpler than that in antiferromagnetic case. The investigations have been conducted in three anisotropy cases, respectively.

### 4.1 $J_z = J < 0$

Without the quantized field ($\Omega = 0$), the energy eigenvalues are

$$E_1^0 = E_3^0 = E_4^0 = J,\ E_2^0 = -3J. \quad (23)$$

and the ground states $|\psi_1^0\rangle$, $|\psi_3^0\rangle$ and $|\psi_4^0\rangle$ are degenerated. The entropy of entanglement of the ground states are $\mathbf{E}(|\psi_3^0\rangle) = 1$, and $\mathbf{E}(|\psi_1^0\rangle) = \mathbf{E}(|\psi_4^0\rangle) = \mathbf{0}$. With the driving field ($\Omega \neq 0$), the ground state is $|\psi_3\rangle$, and the concurrence is given by

$$C = \frac{1}{2} - \frac{\sqrt{(n+1)(n+2)}}{3+2n}, \quad (24)$$

which is independent of the coupling parameters. The mutation from 1 to $C = 1/2 - \sqrt{(n+1)(n+2)}/(3+2n)$ occurs at the critical point $\Omega = 0$ shown as Figure 3a.

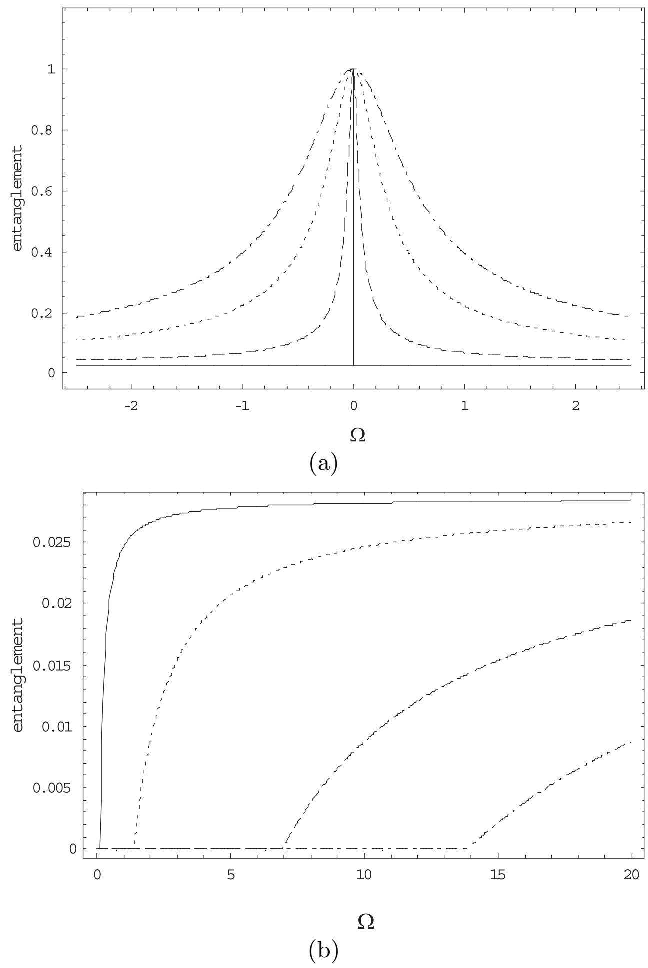


**Fig. 3.** (a) Concurrence of ferromagnetic *XXZ* model versus coupling constant $\Omega$ for different $J$ and $J_z$ ($|J| \geq |J_z|$): $J = J_z = -1$ (solid line), $J = -1$ and $J_z = -0.9$ (dashed line), $J = -1$ and $J_z = -0.5$ (doted line), $J = -1$ and $J_z = 0$ (dash-dot-line). (b) Concurrence of ferromagnetic *XXZ* model versus coupling constant $\Omega$ for different $J$ and $J_z$ ($|J| < |J_z|$): $J = -0.99$ and $J_z = -1$ (solid line), $J = -0.9$ and$J_z = -1$ (doted line), $J = -0.5$ and $J_z = -1$ (dashed line), $J = 0$ and $J_z = -1$ (dash-dot-line).

### 4.2 $J < J_z \leq 0$

When $\Omega \neq 0$, the ground state is $|\psi_3\rangle$ obtained from equation (3). The concurrence is the same as equation (19). The $\Omega$-dependence of the concurrence is shown in Figure 3.

### 4.3 $J_z < J \leq 0$

When $\Omega = 0$, we obtained the eigenvalues from equation (11)

$$E_1^0 = J_z,\ E_2^0 = -J_z - 2J,\ E_3^0 = -J_z + 2J,\ E_4^0 = J_z, \quad (25)$$

ground state is $\left|\psi_1^0\right\rangle$ or $\left|\psi_4^0\right\rangle$ either of which is not entangled state. When $\Omega \neq 0$, eigenvalues become

$$E_1 = J_z,\ E_2 = -J_z - 2J,\ E_3 = J - \zeta,\ E_4 = J + \zeta, \quad (26)$$

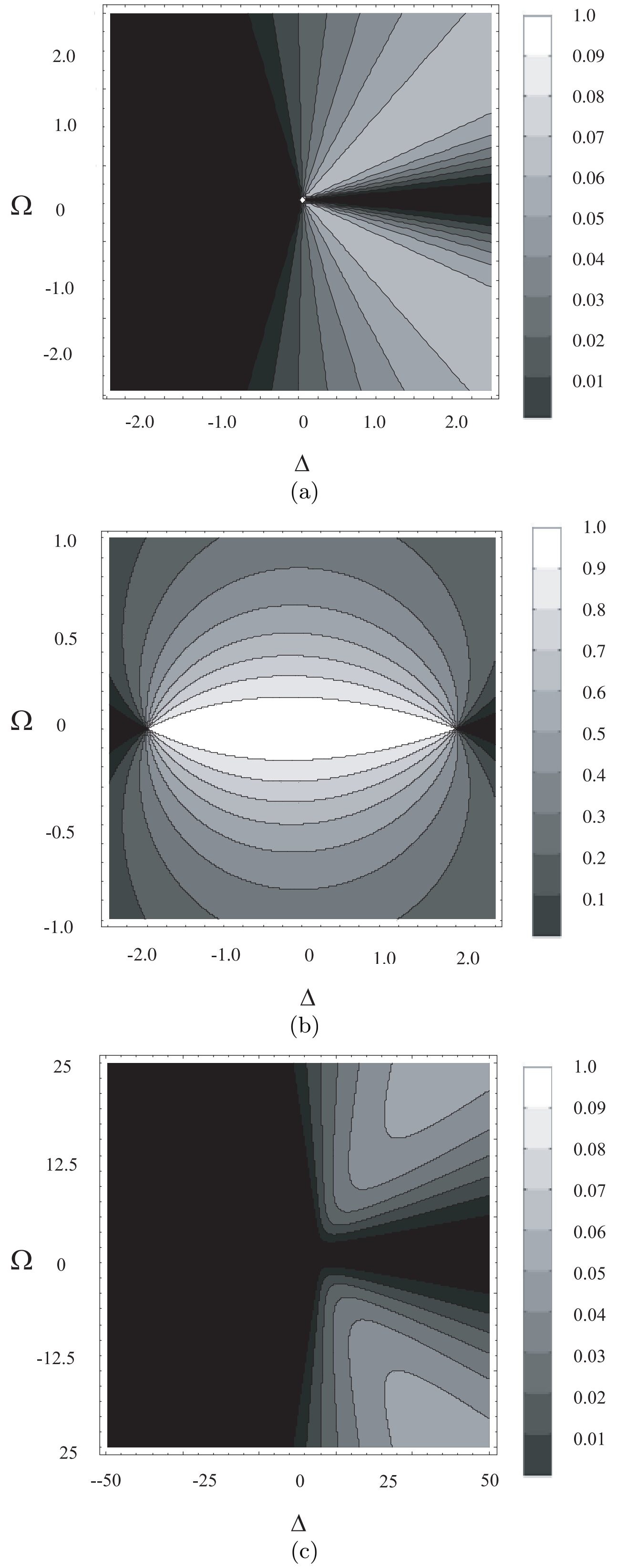


**Fig. 4.** Concurrence of the two-qubit ferromagnetic *XXZ* model versus detuning $\Delta$ ($x$-coordinate) and coupling constant $\Omega$ ($y$-coordinate) for different $J$ and $J_z$. Figures a–c correspond to $J = J_z = -1$, $J = -1$ and $J_z = 0$, $J = 0$ and $J_z = -1$.

with $\zeta = \sqrt{(J_z - J)^2 + 2(3+2n)\Omega^2} > |J_z - J| = (J - J_z)$, the ground state is $|\psi_3\rangle$, and the concurrence is the same as equation (22). The $\Omega$-dependence of concurrence is given in Figure 3 for the resonance case.

Figure 4 shows the contour line of entanglement as a function of detuning $\Delta$ and Rabi frequency $\Omega$. Figure 4b is similar to the antiferromagnetic case (see Fig. 2b, and a mutation of entanglement from 1 to 0 takes place at the detuning $\Delta = 2.0$ coupling constant $\Omega$ close to 0, too.

## 5 Conclusion

In summary, the spin-field coupling dependence ($\Omega$-dependence) of the entanglement in the *XXZ* model possesses a rich structure, sharing in many ways with the properties of QPT. It is similar to the semiclassical case but more complicated properties are observed via purity quantum discussion. The entanglement-denote QPT varies with anisotropic exchange-coupling constants $J$ and $J_z$ for the antiferromagnetic case, which is of fundamental interests in the quantum information and computing.

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